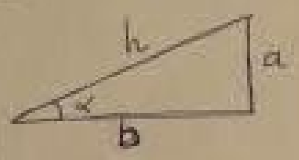
IDENTIDADES TRIGONOMÉTRICAS

-1-



$\operatorname{sen}\alpha = \frac{a}{h}$ $\quad$ $\operatorname{cosec}\alpha = \frac{1}{\operatorname{sen}\alpha} = \frac{h}{a}$

$\cos\alpha = \frac{b}{h}$ $\quad$ $\sec\alpha = \frac{1}{\cos\alpha} = \frac{h}{b}$

$\frac{\operatorname{sen}\alpha}{\cos\alpha} = \operatorname{tg}\alpha = \frac{a}{b}$ = pendiente; $\operatorname{cotg}\alpha = \frac{\cos\alpha}{\operatorname{sen}\alpha} = \frac{1}{\operatorname{tg}\alpha} = \frac{b}{a}$

Relación entre sen y cos

Th Pitágoras: $\operatorname{sen}^2\alpha + \cos^2\alpha = 1$ $\quad$ $\frac{a^2}{h^2} + \frac{b^2}{h^2} = \frac{a^2+b^2}{h^2} = \frac{h^2}{h^2} = 1$ #

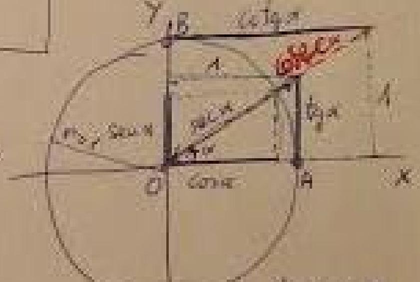
Relación entre sec y tg

$\sec^2\alpha = \frac{1}{\cos^2\alpha} = 1 + \operatorname{tg}^2\alpha$ $\quad$ $1 + \frac{\operatorname{sen}^2\alpha}{\cos^2\alpha} = \frac{\cos^2\alpha + \operatorname{sen}^2\alpha}{\cos^2\alpha} = \frac{1}{\cos^2\alpha}$ #

Relación entre cosec y cotg

$\operatorname{cosec}^2\alpha = \frac{1}{\operatorname{sen}^2\alpha} = 1 + \operatorname{cotg}^2\alpha$

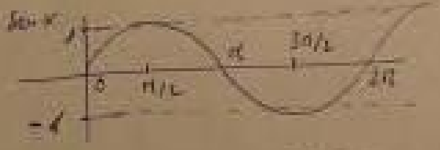
Circunferencia trigonométrica o goniométrica unitaria o unidad

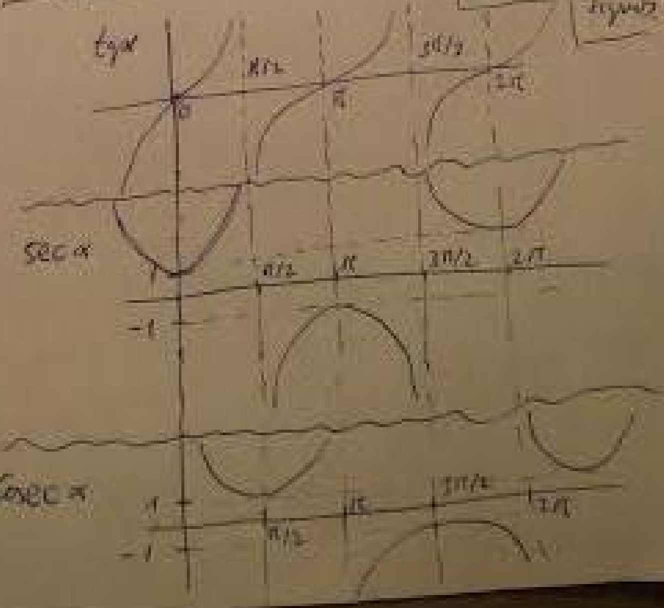


	1º	2º	3º	4º
sen	0→1	1→0	0→-1	-1→0
cos	1→0	0→-1	-1→0	0→1
tg	0→∞	-∞→0	0→+∞	-∞→0
sec	1→∞	-∞→-1	-1→-∞	∞→1
cosec	∞→1	1→∞	-∞→-1	-1→-∞
cotg	∞→0	0→-∞	∞→0	0→-∞

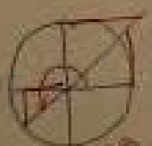
La tg siempre se traza desde A y la cotg desde B

Para la sec y la cosec hay que pensar en los signos





2º cuadrante



3er cuadrante

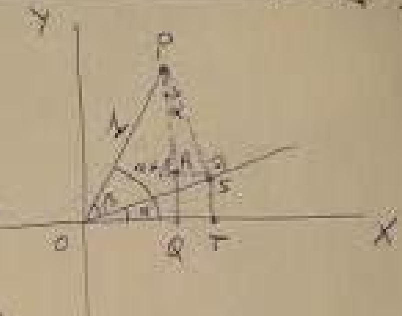
4º cuadrante

sen tg

cos cotg

* **Razones trigonométricas de la suma y diferencia de ángulos**

$$\boxed{\begin{aligned} \operatorname{sen}(\alpha \pm \beta) &= \operatorname{sen}\alpha \cos\beta \pm \operatorname{sen}\beta \cos\alpha \\ \cos(\alpha \pm \beta) &= \cos\alpha \cos\beta \mp \operatorname{sen}\alpha \operatorname{sen}\beta \end{aligned}}$$



$\operatorname{sen}(\alpha+\beta) = PQ = PR + RQ = \operatorname{sen}\alpha \cos\beta + \operatorname{sen}\beta \cos\alpha$ #

$PR = \cos\alpha \operatorname{sen}\beta$

$RQ = \operatorname{sen}\alpha \cos\beta$

Sustituimos β por $-\beta$: $\operatorname{sen}(\alpha-\beta) = \operatorname{sen}\alpha \cos(-\beta) + \operatorname{sen}(-\beta)\cos\alpha = \operatorname{sen}\alpha \cos\beta - \operatorname{sen}\beta \cos\alpha$ #

$\cos(\alpha+\beta) = OQ = OT - QT = \cos\alpha \cos\beta - \operatorname{sen}\alpha \operatorname{sen}\beta$ #

$OT = \cos\alpha \cos\beta$

$QT = \operatorname{sen}\alpha \operatorname{sen}\beta$

Sustituimos β por $-\beta$: $\cos(\alpha-\beta) = \cos\alpha \cos(-\beta) + \operatorname{sen}\alpha \operatorname{sen}(-\beta) = \cos\alpha \cos\beta - \operatorname{sen}\alpha \operatorname{sen}\beta$ #

$$tg(\alpha \pm \beta) = \frac{\operatorname{sen}(\alpha \pm \beta)}{\cos(\alpha \pm \beta)} = \frac{\operatorname{sen}\alpha \cos\beta \pm \operatorname{sen}\beta \cos\alpha}{\cos\alpha \cos\beta \mp \operatorname{sen}\alpha \operatorname{sen}\beta} \overset{\text{dividimos entre } \cos\alpha \cos\beta}{=}$$

$$= \frac{\frac{\operatorname{sen}\alpha \cancel{\cos\beta}}{\cos\alpha \cancel{\cos\beta}} \pm \frac{\operatorname{sen}\beta \cancel{\cos\alpha}}{\cancel{\cos\alpha} \cos\beta}}{\frac{\cancel{\cos\alpha \cos\beta}}{\cancel{\cos\alpha \cos\beta}} \mp \frac{\operatorname{sen}\alpha \operatorname{sen}\beta}{\cos\alpha \cos\beta}} \;; \quad \boxed{tg(\alpha \pm \beta) = \frac{tg\,\alpha \pm tg\,\beta}{1 \mp tg\,\alpha \cdot tg\,\beta}}$$

* **Razones trigonométricas del ángulo doble**: Haciendo en las anteriores $\alpha = \beta$:

$$\boxed{\begin{aligned} \operatorname{sen} 2\alpha &= 2 \operatorname{sen}\alpha \cos\alpha \\ \cos 2\alpha &= \cos^2\alpha - \operatorname{sen}^2\alpha \\ tg\, 2\alpha &= \frac{2\, tg\,\alpha}{1 - tg^2\alpha} \end{aligned}}$$

* **Razones trigonométricas del ángulo mitad**: Haciendo en las anteriores $\alpha = \frac{\alpha}{2} \Rightarrow \operatorname{sen}\alpha = 2 \operatorname{sen}\frac{\alpha}{2} \cos\frac{\alpha}{2}$. Para mejor la del cos

$$\cos\alpha = \cos^2\frac{\alpha}{2} - \operatorname{sen}^2\frac{\alpha}{2} = \cos^2\frac{\alpha}{2} - 1 + \cos^2\frac{\alpha}{2} =$$

$$= 2\cos^2\frac{\alpha}{2} - 1 \qquad \boxed{\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}}$$

- ⓪ también: $\cos\alpha = 1 - \operatorname{sen}^2\frac{\alpha}{2} - \operatorname{sen}^2\frac{\alpha}{2} = 1 - 2\operatorname{sen}^2\frac{\alpha}{2}$

$2\operatorname{sen}^2\frac{\alpha}{2} = 1 - \cos\alpha \quad \therefore \quad \boxed{\operatorname{sen}\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}}$

$\operatorname{tg}\frac{\alpha}{2} = \dfrac{\operatorname{sen}\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \qquad \boxed{\operatorname{tg}\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}}$

* Transformaciones de sumas en productos

Partimos de: $\begin{cases} \operatorname{sen}(\alpha+\beta) = \operatorname{sen}\alpha\cos\beta + \operatorname{sen}\beta\cos\alpha \\ \operatorname{sen}(\alpha-\beta) = \operatorname{sen}\alpha\cos\beta - \operatorname{sen}\beta\cos\alpha \end{cases}$

Sustituimos

Hacemos $\alpha+\beta = A$, $\alpha-\beta = B$ de forma que $\begin{cases} \alpha = \frac{A+B}{2} \\ \beta = \frac{A-B}{2} \end{cases}$

→ Sumamos $\boxed{\operatorname{sen}A + \operatorname{sen}B = 2\operatorname{sen}\frac{A+B}{2}\cos\frac{A-B}{2}}$

→ Restamos $\boxed{\operatorname{sen}A - \operatorname{sen}B = 2\cos\frac{A+B}{2}\operatorname{sen}\frac{A-B}{2}}$

Sustituimos

Hacemos igual para el coseno:

$\begin{cases} \cos(\alpha+\beta) = \cos\alpha\cos\beta - \operatorname{sen}\alpha\operatorname{sen}\beta \\ \cos(\alpha-\beta) = \cos\alpha\cos\beta + \operatorname{sen}\alpha\operatorname{sen}\beta \end{cases}$

→ Sumamos $\boxed{\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}$

→ Restamos $\boxed{\cos A - \cos B = -2\operatorname{sen}\frac{A+B}{2}\operatorname{sen}\frac{A-B}{2}}$

* Transformación de productos en sumas.

Sustituimos en las cuatro anteriores $\frac{A+B}{2}=\alpha$ y $\frac{A-B}{2}=\beta$

con lo que $\begin{cases} A=\alpha+\beta \\ B=\alpha-\beta \end{cases}$

$$\boxed{\begin{aligned} \operatorname{sen}\alpha\cdot\cos\beta &= \frac{1}{2}\left[\operatorname{sen}(\alpha+\beta)+\operatorname{sen}(\alpha-\beta)\right] \\ \cos\alpha\cdot\operatorname{sen}\beta &= \frac{1}{2}\left[\operatorname{sen}(\alpha+\beta)-\operatorname{sen}(\alpha-\beta)\right] \\ \cos\alpha\,\cos\beta &= \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right] \\ \operatorname{sen}\alpha\,\operatorname{sen}\beta &= \frac{-1}{2}\left[\cos(\alpha+\beta)-\cos(\alpha-\beta)\right] \end{aligned}}$$

Identidad de Euler/Lindemann

$z=\pi \Rightarrow \boxed{e^{i\pi}+1=0}$

Conviene a decir en el plano complejo que $\cos\pi=-1$

* Sen, cos, tg de los ángulos más comunes:

	$\pi/6$ 30°	$\pi/4$ 45°	$\pi/3$ 60°
sen	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$
cos	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$
tg	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

* Serie de Taylor

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

* Serie de Maclaurin (Taylor centrado en cero) $a=0$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n \Rightarrow$$

$$\boxed{e^{zi}=\cos z+i\operatorname{sen} z}$$

$\Rightarrow$ Para $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\dots=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

$$\operatorname{sen}x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\pm\dots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}\pm\dots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

Sustituimos x por zi: $e^{zi}=1+\frac{zi}{1!}-\frac{z^2}{2!}-\frac{z^3 i}{3!}+\frac{z^4}{4!}+\frac{z^5 i}{5!}\pm\dots=$

$$=\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\dots\right)+\left(\frac{z}{1!}-\frac{z^3}{3!}+\frac{z^5}{5!}-\dots\right)i=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}+i\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!}$$